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PDA-BCJR Algorithm for Factorial Hidden Markov Models with Application to MIMO Equalisation

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Outline of the talk

- MIMO communications problem
- Graphical Models – introduction, application, types, problems in communications, inference algorithms
- proposed PDA-BCJR algorithm
- Numerical results
- Conclusions

Probabilistic Graphical Models - PGM

- **PGMs apply to decision making and/or estimation in the presence of uncertainty**

- **PGMs represent families of probability distribution functions**

- **PGMs do not provide solutions on its own, but can provide:**
  - new insight into existing models
  - motivation for new models
  - suggest construction of new algorithms
  - unified view of problems from smilingly different disciplines of science

Where they are applied and types

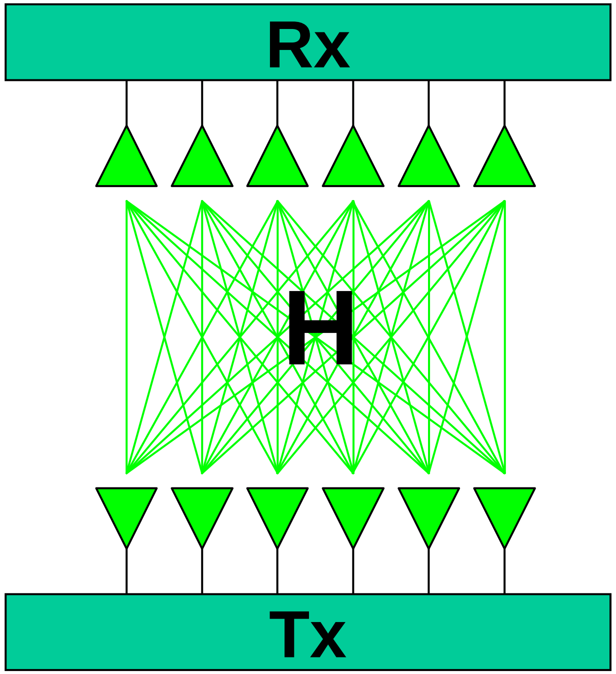
- **Applied (and often independently developed) in:**
  - Bio-informatics (Bio-statistics),
  - Machine Learning (neural nets)
  - Speech processing and image processing
  - Communications, Information retrieval
  - Forensic science
  - and many more

- **There are three most common types of graphs:**
  - Directed acyclic graphs (DAG),
  - Undirected graphs (UG)
  - Factor graphs (FG)

Algorithms for inference in PGMs

- **Exact:**
  - Kalman Filter/Smoother
  - Forward-Backward
  - Sum-product
  - Junction-Tree algorithm (supersedes the above)
- **Monte Carlo (Sampling)**
  - Direct Sampling, Importance sampling
  - MCMC: Gibbs, Metropolis-Hastings
  - Sequential: Particle Filters/Smoothers
- **Deterministic Approximations**
  - Variational Approximation, EM and its variants
  - PDA, Expectation Propagation, GPB, etc..

Considered problem – MIMO communications



It is so called wideband system so the channels are modelled as multi-dimensional FIR filters:

$$\mathbf{H} = \begin{bmatrix} \mathbf{h}_{1,1}^T & \cdots & \mathbf{h}_{1,N}^T \\ \vdots & \ddots & \vdots \\ \mathbf{h}_{M,1}^T & \cdots & \mathbf{h}_{M,N}^T \end{bmatrix}$$

$$\mathbf{h}_{m,n} = \left(h_{m,n}^{(\tau=0)}, h_{m,n}^{(\tau=1)}, \ldots, h_{m,n}^{(\tau=L-1)} \right)^T$$

Generating model

$$\mathbf{y}^{(t)} = \mathbf{H}\mathbf{x}^{(t)} + \mathbf{n}^{(t)}$$

$$\mathbf{n}^{(t)} \sim \mathcal{N}\left(0, \sigma_n^2 \mathbf{I}\right)$$

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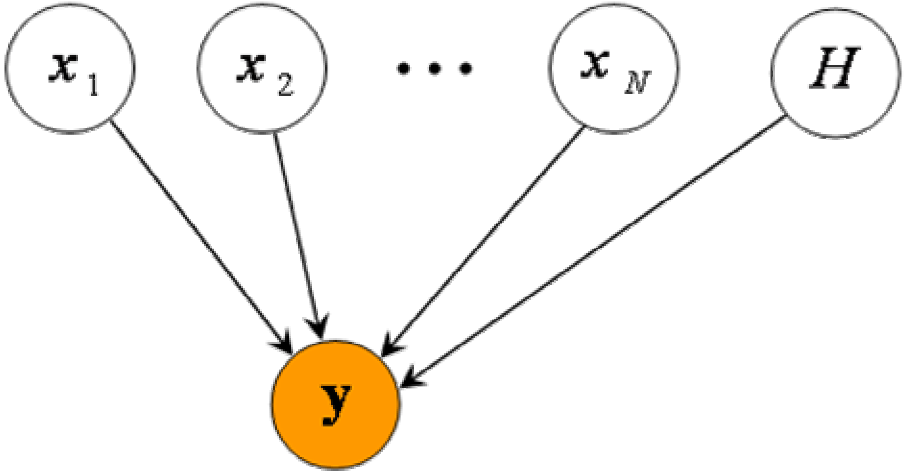
DAG Factorial Model

Communications in non-orthogonal channels: CDMA, MIMO



$$\mathbf{y} = \sum_i \mathbf{h} x_i + \mathbf{n}$$

In our case, **x** – are discrete variables and a marginal of **y** is a mixture Gaussian

$$f(\mathbf{y}, x_1, \ldots, x_N, \mathbf{H}) = f(\mathbf{y} \mid x_1, \ldots, x_N, \mathbf{H})\, f(\mathbf{H}) \prod_{i=1}^{N} f(x_i)$$

Task:

$$f(x_i \mid \mathbf{y}) = \int_{\mathbf{H}} \sum_{x_{-i}} f(x_i, x_{-i}, \mathbf{H} \mid \mathbf{y})\, d\mathbf{H} =$$

$$\int_{\mathbf{H}} \sum_{x_{-i}} f(x_i \mid x_{-i}, \mathbf{H}, \mathbf{y})\, f(x_{-i} \mid \mathbf{H}, \mathbf{y})\, f(\mathbf{H} \mid \mathbf{y})\, d\mathbf{H}$$

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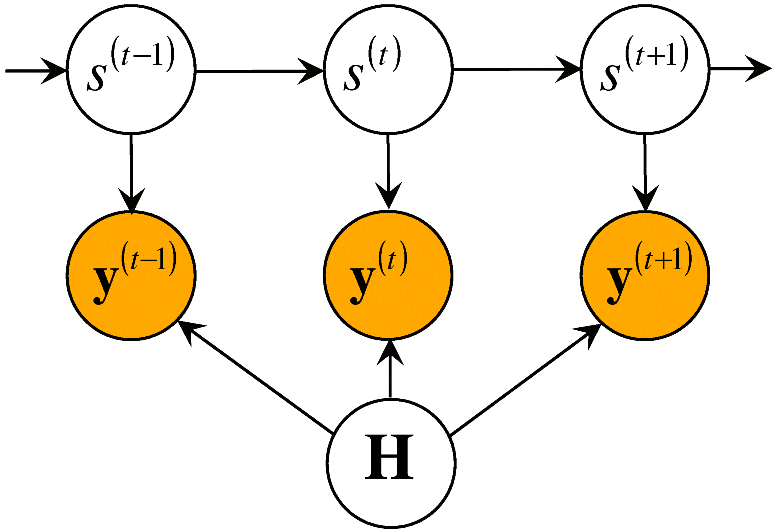
DAG Hidden Markov Model

Received signal in channels with memory, convolutionally encoded signal (binary codes, space-time trellis codes)



$$f\left(\mathbf{y}^{(1)},\ldots,\mathbf{y}^{(T)},s^{(1)},\ldots,s^{(T)},\mathbf{H}\right)=$$
$$f(\mathbf{H})\prod_{t=1}^{T} f\left(\mathbf{y}^{(t)}\middle|s^{(t)},\mathbf{H}\right) f\left(s^{(t)}\middle|s^{(t-1)}\right)$$

States: $s^{(t)} \equiv \left\{x^{(t)},x^{(t-1)},\ldots,x^{(t-L+1)}\right\}$

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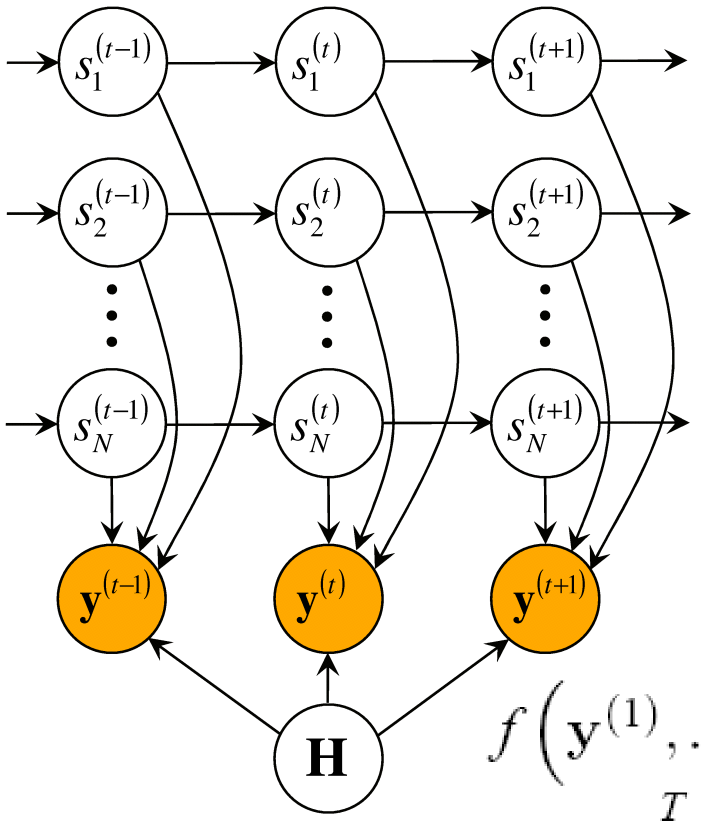
DAG Factorial Hidden Markov Model

Received multiuser (or/and multi-antenna i.e. MIMO) signal in channels with memory



Overall FHMM model arises by replacing the single random variables with HMM

$$f\left(\mathbf{y}^{(1)},\ldots,\mathbf{y}^{(T)},s_1^{(1)},\ldots,s_1^{(T)},\ldots,s_N^{(1)},\ldots,s_N^{(T)},\mathbf{H}\right) =$$
$$f(\mathbf{H})\prod_{t=1}^{T}\left[f\left(\mathbf{y}^{(t)}\middle|s_1^{(t)},\ldots s_N^{(t)},\mathbf{H}\right)\prod_{n=1}^{N}f\left(s_n^{(t)}\middle|s_n^{(t-1)}\right)\right]$$

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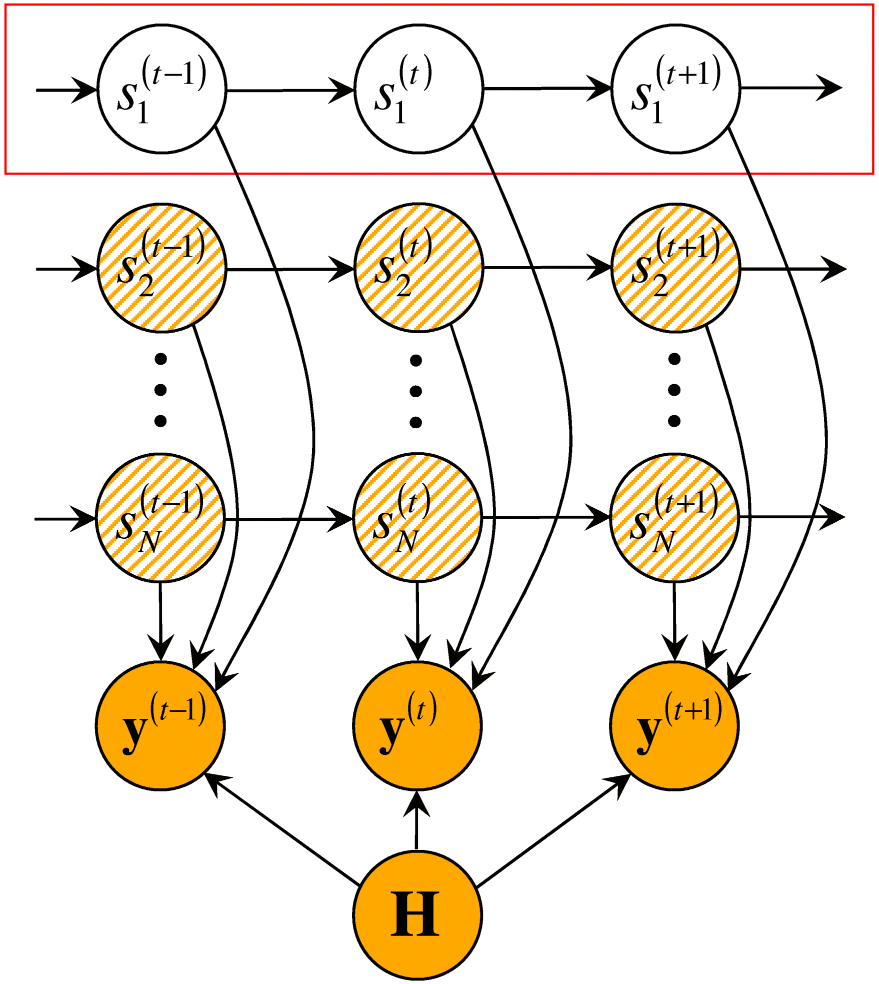
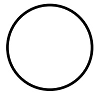
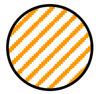
PDA – BCJR Algorithm

Iterations on the chains

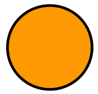




Latent variable (uncertainly fully accounted for)



Latent variable (uncertainly partially accounted for via Gaussian approximation)



Observed variable

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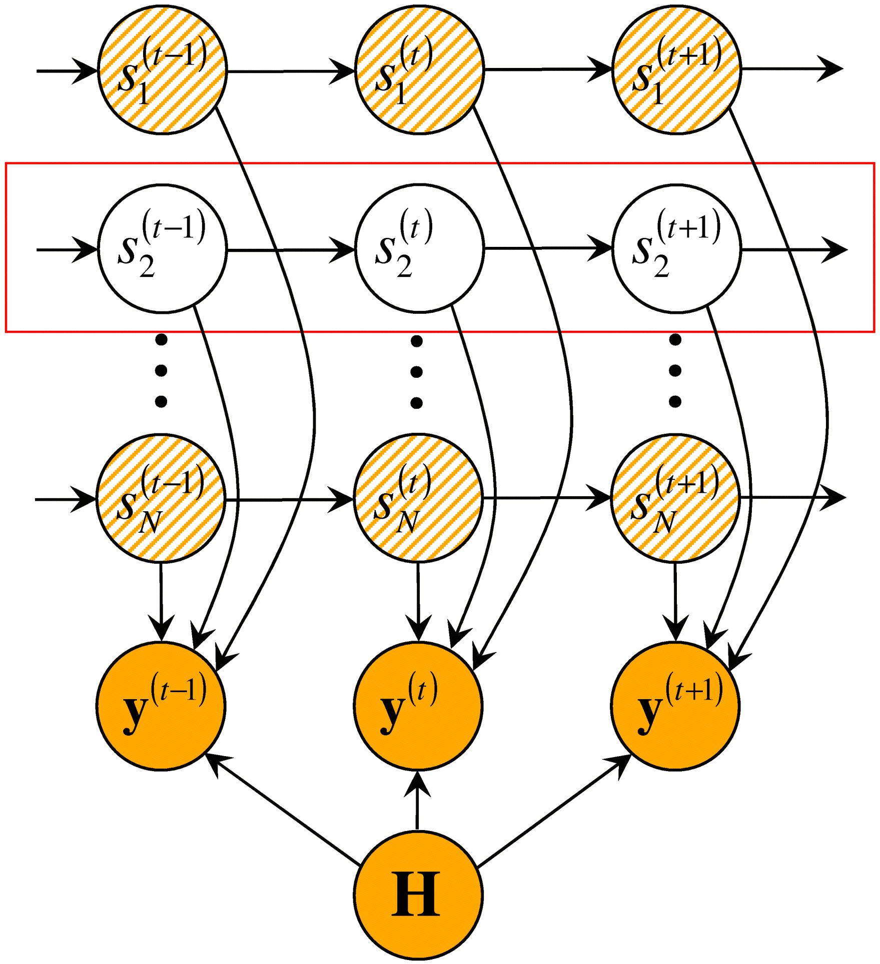
PDA – BCJR Algorithm

Iterations on the chains



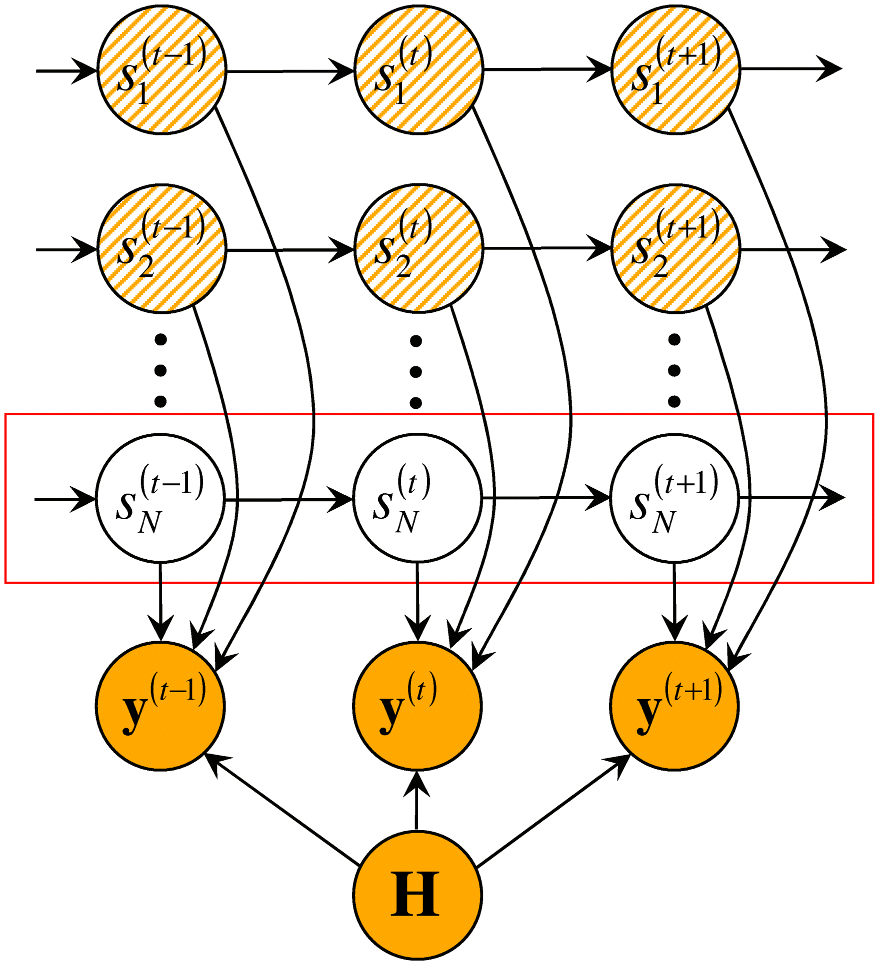
Latent variable (uncertainly fully accounted for)

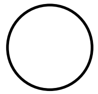
Latent variable (uncertainly partially accounted for via Gaussian approximation)

Observed variable

PDA – BCJR Algorithm

Iterations on the chains





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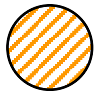

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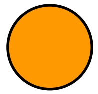
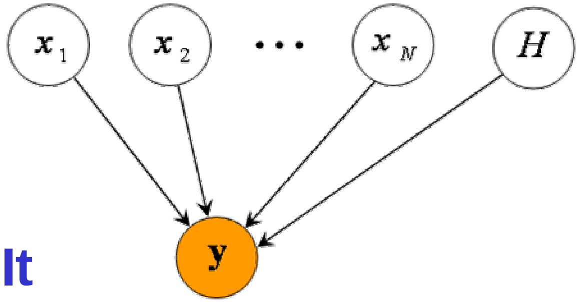
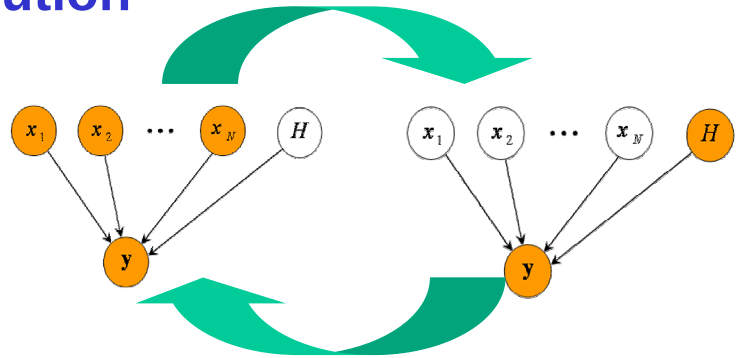


Observed variable

EM – accounting for channel uncertainty



EM solves the "chicken and egg" problem. It is useful where the underlying distribution has a form:



$$f\left(\mathbf{Y}\middle|\theta\right)=\int_X f\left(\mathbf{Y},X\middle|\theta\right)$$

"Chicken and egg"

Set up as estimation (ML or MAP) of **H**, where **X** is the missing data. I.e. we are settling for the expectation of the latent data, rather than the data itself

$$Q\left(\mathbf{H},\mathbf{H}^{(i)}\right)=\underset{\mathbf{x}|\mathbf{H}^{(i)},\mathbf{y}}{E}\left\{\log\left(f\left(\mathbf{H},\mathbf{x}\middle|\mathbf{y}\right)\right)\middle|\mathbf{H}^{(i)},\mathbf{y}\right\}$$

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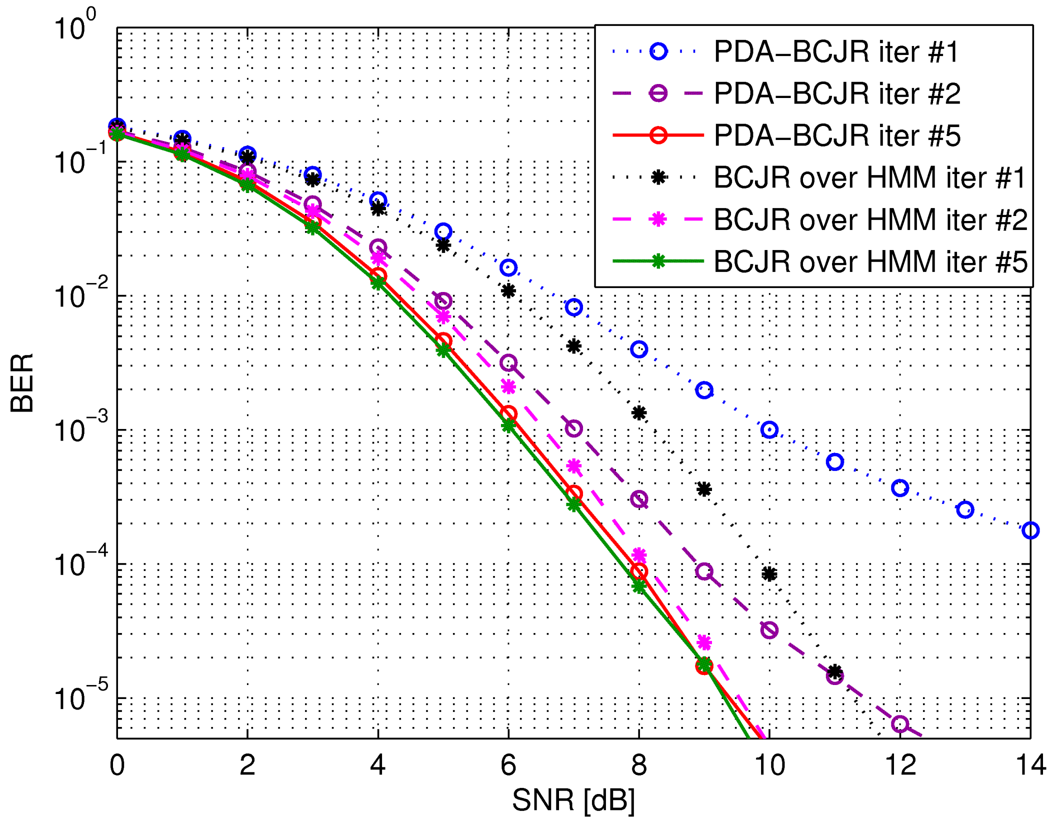
PDA-BCJR results I

- PDA-BCJR
- Structured Variational
- Exact (BCJR over HMM)

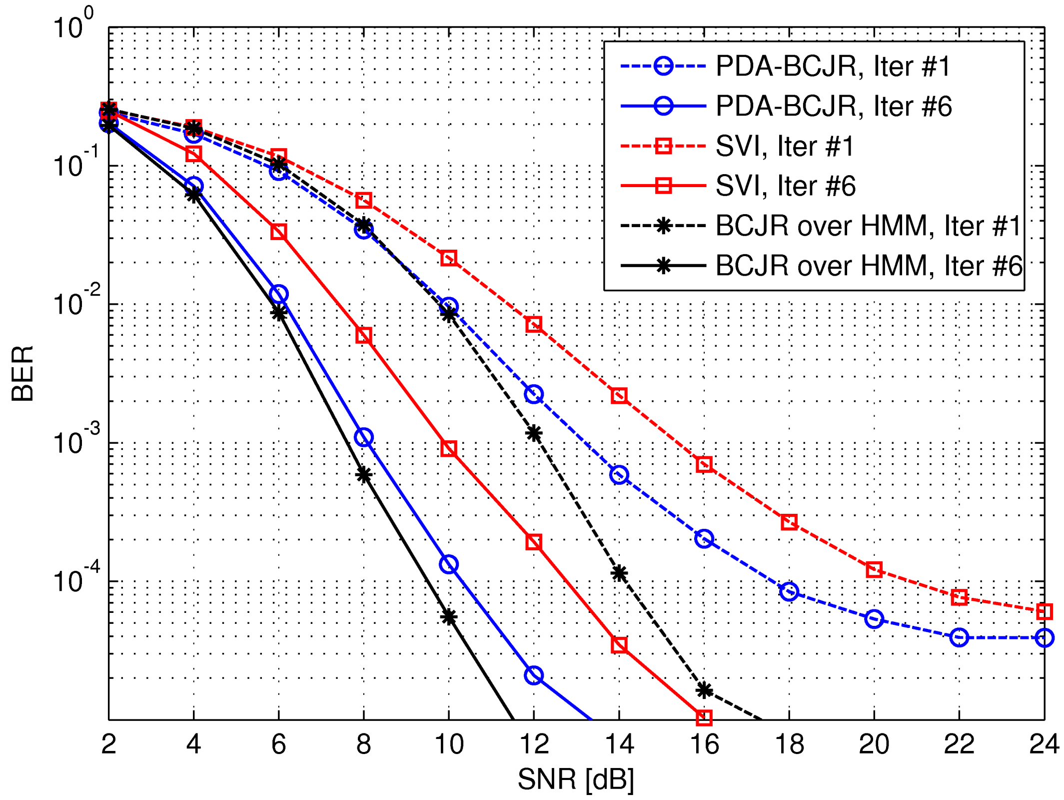
MIMO System with NT=NR=3 antennas, BPSK, 3 tap channel, Channels perfectly known



PDA-BCJR results II – "semiblind setting"

- PDA-BCJR EM
- Structured Variational EM
- BCJR over HMM EM

MIMO System with NT=NR=3 antennas, BPSK, 3 tap channel, first 18 symbols known.



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Conclusions

- A generalisation to PDA has been proposed – iterations on entire HMMs
- Particularly suitable in conjunction with EM
- Further degrees of approximation are possible (e.g. within the chains)
- Another application: speech recognition

Additional slides I

Table 1: The PDA-BCJR algorithm summary

"'Outer recursions"'
for $I = 1 : I_{PDAmax}$
"'PDA recursions"'
for $n = 1 : N$
$\mu_{n,in} = \sum_{j=1, j\neq n}^{N} \mu_{j,out}; \quad \mathbf{\Sigma}_{n,in} = \sum_{j=1, j\neq n}^{N} \mathbf{\Sigma}_{j,out}$
$\left[f\left(x_n^{(t),t=1:T} \mid \mathbf{y}\right), \mu_{n,out}, \mathbf{\Sigma}_{n,out} \right] = BCJR^* \left(\mathbf{y}, \hat{\mathbf{H}}_{:,n}, f\left(x_n^{(t),t=1:T}\right), \mu_{n,in}, \mathbf{\Sigma}_{n,in} \right)$
end, end

$$f\left(s', s \mid \mathbf{y}\right) \propto \underbrace{f\left(s', \mathbf{y}^{(1:t-1)}\right)}_{\alpha^{(t)}} \underbrace{f\left(s \mid s'\right) f\left(\mathbf{y}^{(t)} \mid s', s\right)}_{\gamma^{(t)}} \underbrace{f\left(\mathbf{y}^{(t+1:T)} \mid s\right)}_{\beta^{(t)}}$$

$$\gamma^{*(t)}\left(s', s\right) \propto \exp\left(-\left(\mathbf{z}_n^{(t)}\right)^H \left(\mathbf{\Sigma}_{n,in}^{(t)}\right)^{-1} \left(\mathbf{z}_n^{(t)}\right) \right) f\left(x_n\right)$$

Where:

$$\mathbf{z}_n^{(t)} = \mathbf{y}^{(t)} - \hat{\mathbf{H}}_{(:,n)} \mathbf{x}_n - \mu_{n,in}^{(t)}$$

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Additional slides II

The output moments are calculated as

$$\mu_{n,out}^{(t)} = E\left\{\hat{\mathbf{H}}_{:,n}\mathbf{x}_n\right\} = \sum_{j=1}^{S}\sum_{k=1}^{K}\hat{\mathbf{H}}_{:,n}\mathbf{x}_n f\left(s', s\,|\mathbf{y}\right)$$

$$\begin{aligned}\boldsymbol{\Sigma}_{n,out}^{(t)} &= E\left\{\left(\hat{\mathbf{H}}_{:,n}\mathbf{x}_n - \mu_{n,out}^{(t)}\right)\left(\hat{\mathbf{H}}_{:,n}\mathbf{x}_n - \mu_{n,out}^{(t)}\right)^H\right\} \\ &= \sum_{j=1}^{S}\sum_{k=1}^{K}\left(\hat{\mathbf{H}}_{:,n}\mathbf{x}_n - \mu_{n,out}^{(t)}\right)\left(\hat{\mathbf{H}}_{:,n}\mathbf{x}_n - \mu_{n,out}^{(t)}\right)^H f\left(s', s\,|\mathbf{y}\right)\end{aligned}$$

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Additional slides III

Connections between Variational Inference and PDA

$$f_{PDA}(x_i = a_\omega | \mathbf{y}) \overset{appox}{\propto} \exp\left((\psi - \mathbf{h}_i a_\omega)^T \mathbf{M}^{-1} (\psi - \mathbf{h}_i a_\omega) \right)$$

where:

$$\mathbf{M} = \frac{\sigma^2}{2}\mathbf{I} + \sum_{j=1, j \neq i}^{N} \mathbf{h}_j^T \mathbf{h}_j Var\{x_j\}$$

The relation is:

$$\mathbf{M}_{PDA} = \mathbf{M}_{VI} + \sum_{j=1, j \neq i}^{N} \mathbf{h}_j^T \mathbf{h}_j Var\{x_j\}$$

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